

# Uncertainty in multi-scale fatigue life modeling and a new approach to estimating frequency of in-service inspection of aging components<sup>1</sup>

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**Abstract.** Uncertainty in modeling the fatigue life of a full-scale component using experimental data at microscopic (Level 1), specimen (Level 2), and full-size (Level 3) scales, is addressed by applying statistical theory of prediction intervals, and that of tolerance intervals based on the concept of coverage,  $p$ . Using a nonlinear least squares fit algorithm and the physical assumption that the one-sided Lower Tolerance Limit (*LTL*), at 95% confidence level, of the fatigue life, i.e., the minimum cycles-to-failure,  $\min N_f$ , of a full-scale component, *cannot be negative* as the lack or “Failure” of coverage ( $Fp$ ), defined as  $1 - p$ , approaches zero, we develop a new fatigue life model, where the minimum cycles-to-failure,  $\min N_f$ , at extremely low “Failure” of coverage,  $Fp$ , can be estimated. Since the concept of coverage is closely related to that of an inspection strategy, and if one assumes that the predominant cause of failure of a full-size component is due to the “Failure” of inspection or coverage, it is reasonable to equate the quantity,  $Fp$ , to a Failure Probability,  $FP$ , thereby leading to a new approach of estimating the frequency of in-service inspection of a full-size component. To illustrate this approach, we include a numerical example using the published data of the fatigue of an AISI 4340 steel (N.E. Dowling, *Journal of Testing and Evaluation*, ASTM, Vol. 1(4) (1973), 271–287) and a linear least squares fit to generate the necessary uncertainties for performing a dynamic risk analysis, where a graphical plot of an estimate of risk with uncertainty vs. a predicted most likely date of a high consequence failure event becomes available. In addition, a nonlinear least squares logistic function fit of the fatigue data yields a prediction of the statistical distribution of both the ultimate strength and the endurance limit.

**Keywords:** Aging component, AISI 4340 steel, coverage, dynamic risk analysis, endurance limit, failure of coverage, failure probability, fatigue life modeling, fracture mechanics, full-scale component, in-service inspection, least squares fit, logistic function, multi-scale, nonlinear least squares fit, nuclear powerplant, prediction intervals, risk-informed inspection strategy, statistical analysis, tolerance intervals, ultimate tensile strength, uncertainty quantification

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## 1. Introduction

The failure of a complex engineering structure, such as a long-span suspension bridge, or a simple component such as an aircraft window, has a common feature, namely the initiation and propagation of one or more microscopic discontinuities such as voids, micro-cracks, etc. For example, in Fig. 1, we show three cases of such feature, where the initiation and growth of a micro-crack in a steel specimen were documented by Nisitani et al. [1], the distribution of micro-void density as a function of void diameter at a fixed time during a fatigue test of an irradiated steel specimen was reported by Mercks [2], and the frequency of micro-cracks as a function of crack length during three stages of a corrosion fatigue test of a steel specimen was exhibited by Kitagawa and Suzuki [3].

Any one of those three experiments involved microscopy, which, for convenience, will be referred to in this paper as “Level 1 – Micro”. All three examples were also associated with one or more test specimens of laboratory scale, as shown in Figs 2 and 3, where we introduce higher levels of scale such as “Level 2 – Specimen”, “Level 3 – Component”, etc.

Based on the statistical theory of a “prediction interval”, any fatigue life model using data from a finite number,  $n$ , of specimens at either Level 1 or 2, is *only* capable of predicting life at a certain confidence,  $(1 - \alpha)$  100%, for an infinitely large population at either Level 1 or 2, respectively. To illustrate the limitation of a prediction interval at, say, the specimen Level 2, let us consider a cycles-to-failure prediction at Level 2 to be at a 95% confidence level, i.e.,  $(1 - \alpha)$  100% = 95%, or,  $\alpha = 0.05$ . As shown by Nelson, Coffin, and Copeland [4, pp. 179–180], when the true mean,  $\mu$ , and standard deviation,  $\sigma$ , of a normal distribution are not known, the so-called  $(1 - \alpha)$  100% prediction interval is given by the following expression:

$$\bar{y} \pm t(\alpha/2; n - 1)s\sqrt{1 + \frac{1}{n}}, \quad (1)$$

where  $\bar{y}$  is the estimated mean,  $s$ , the estimated standard deviation,  $n$ , the sample size,  $t$ , the well-known Student’s distribution function, and  $\alpha$ , the quantity associated with the confidence level given by  $(1 - \alpha)$  100%. For engineers dealing with experimental data at the specimen Level 2, the estimated prediction interval given in equation (1) for a normally distributed sample data is valid only at the Level 2 scale, and not at a higher level such as Level 3, the level of a full-size component. In short, *a prediction interval is only valid for a single-scale model.*

To extrapolate a Level 2 estimate to that of a higher level, we need to introduce a new concept, i.e., the concept of “coverage”,  $p$ , which is defined as the proportion of the population that is covered by a new statistical interval known as the “tolerance interval”, (see, e.g., Nelson et al. [4, pp. 179–180]). The upper limit and lower limit of the tolerance interval are known as the upper tolerance limit (*UTL*) and lower tolerance limit (*LTL*), respectively. It is the one-sided *LTL* for a given coverage,  $p$ , and the  $(1 - \alpha)$  100% confidence level that engineers are most interested in, whether it is for finding a code-allowable minimum strength of a material for structural design, or the minimum cycles-to-failure,  $\min Nf$ , of a material for a rotary equipment. The reason for choosing the one-sided *LTL* to work with is that the statistical quantity called the confidence level,  $\gamma$ , or,  $(1 - \alpha)$ , is commonly associated with engineering reliability, which is a safety concept based on the assumed existence of a minimum strength of a structure, or, in the case of fatigue life design, a minimum cycles-to-failure,  $\min Nf$ .

The theory of one-sided or two-sided tolerance intervals for a normal population is well-established in the statistics literature (see, e.g., Prochan [5], Natrella [6], and Nelson et al. [4]). For example, as shown by Nelson et al. [4], the tolerance interval of fatigue life,  $Nf_3$ , for an infinitely large normal population of full-scale components at Level 3, can be expressed in terms of the estimated sample mean,  $\bar{y}$ , or,  $Nf_2$ , and

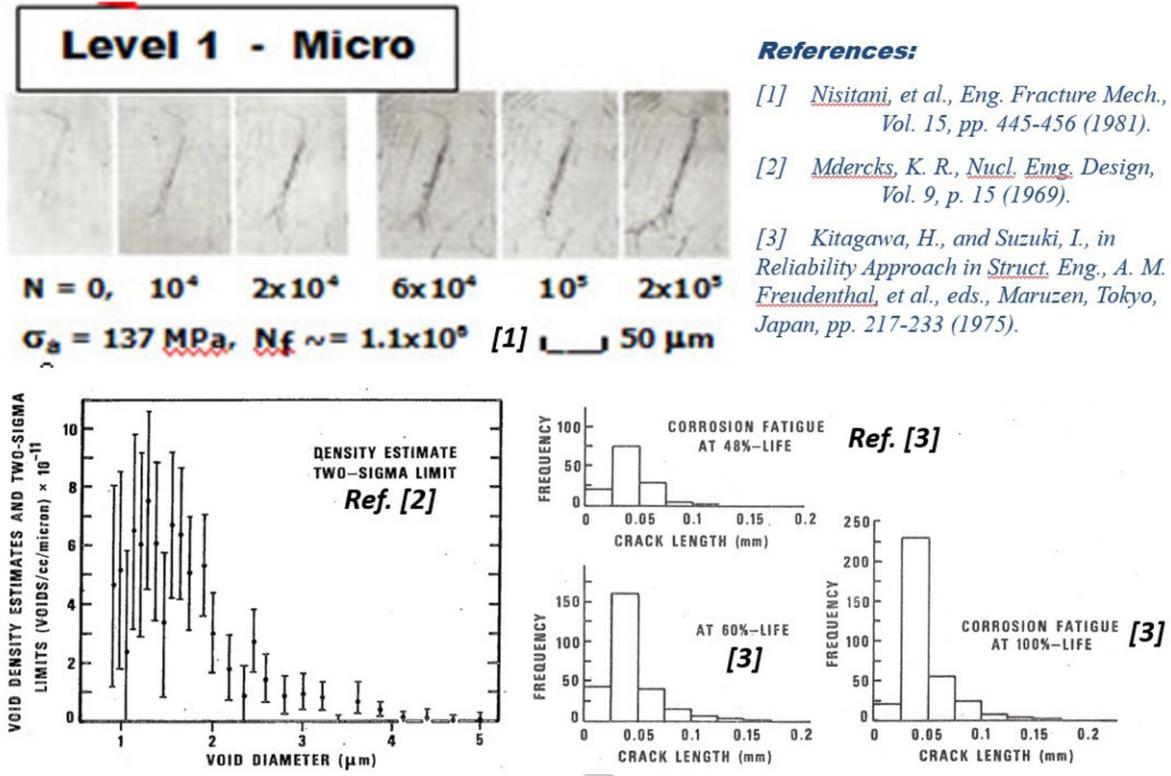


Fig. 1. Fatigue and fracture measurements at the microscopic level [1–3].

the sample standard deviation,  $s$ , or  $sdNf_2$ , of the experimental data derived from  $n$  specimens at Level 2, as shown below:

$$Nf_3 = \bar{y} \pm rus, \tag{2}$$

where  $\bar{y} = Nf_2$ ,  $s = sdNf_2$ , the factor,  $r(n, p)$ , depends on the sample size,  $n$ , and the coverage,  $p$ , and the factor,  $u(df, \gamma)$ , depends on the degrees of freedom,  $df$ , defined by  $n - 1$ , and the confidence level,  $\gamma$ , defined by  $1 - \alpha$ .

Both factors of  $r$  and  $u$  in equation (2) for a normal population are available for a broad range of  $n$ ,  $p$ , and  $\gamma$ , in tabular forms in many statistics books such as Natrella [6] and Nelson et al. [4]. Unfortunately, Nelson et al. [4] gives only tables of the two-sided *LTL*, whereas Natrella [6] gives both two-sided and one-sided *LTL*. As mentioned earlier, for engineering applications, it is the one-sided *LTL* that is of interest, so in this paper, we will only use tables from Natrella [6] to develop a multi-scale fatigue life model where the uncertainty in the fatigue life,  $Nf_3$ , at the full-size component level (Level 3) is quantified by applying the one-sided *LTL* formula of equation (2) using the mean fatigue life,  $Nf_2$ , and its standard deviation,  $sdNf_2$ , as computed from data at the specimen level (Level 2).

In Sections 2 through 6, we describe the theory of a multi-scale fatigue life model in five steps, as listed below, together with a numerical example illustrating each of the five steps.

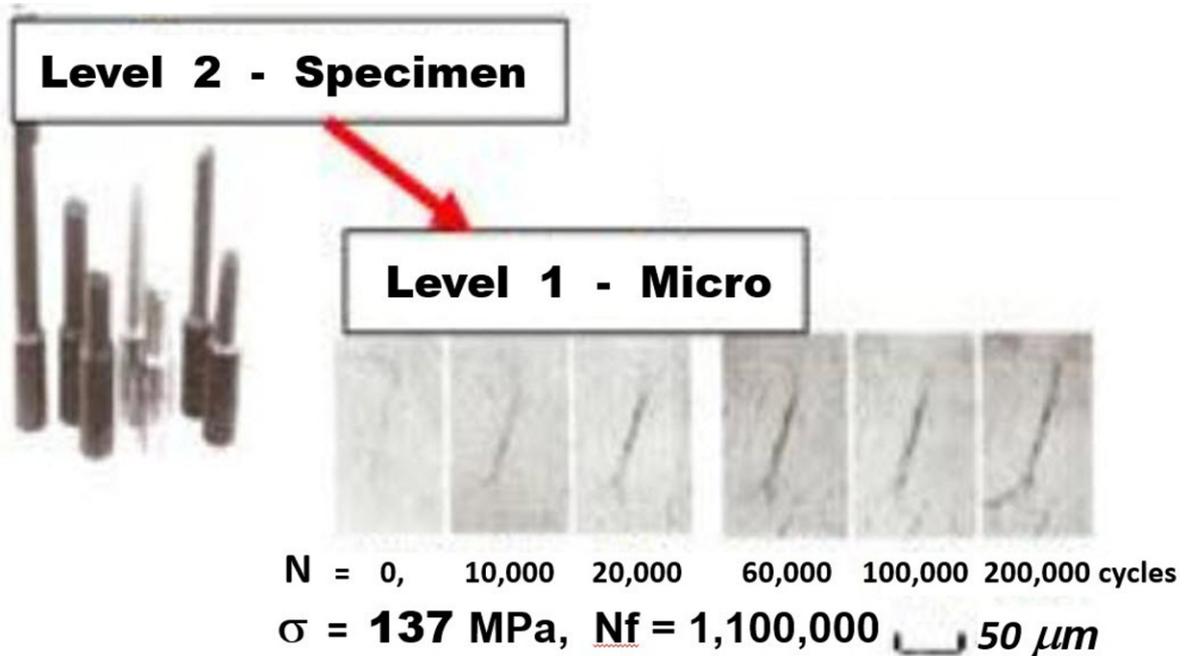


Fig. 2. Fatigue measurements at the microscopic level [1] vs. sizes of fatigue specimens.

- Step 1:** *Level 2 Model.* Identify a fatigue life model formula at the specimen level (Level 2).
- Step 2:** *Level 2 Life vs. Stress.* Run fatigue experiments to obtain cycles-to-failure,  $Nf_2$ , as a function of the applied stress amplitude,  $\sigma_a$ , or, in the absence of available experimental data, compute  $Nf_2$  using the formula identified in Step 1 with the parameters in the formula estimated from either available experimental data or handbook values at specimen Level 2.
- Step 3:** *Level 2 Life with Uncertainty Quantification at Operating Stress.* Use the linear least squares fit algorithm to obtain a log–log plot of  $Nf_2$  vs.  $\sigma_a$ , and obtain, for some operating stress amplitude,  $(\sigma_a)_{op}$ , an estimate of the predicted fatigue life,  $(Nf_2)_{op}$ , and its standard deviation,  $(sdNf_2)_{op}$ .
- Step 4:** *Level 3 Life with Uncertainty Quantification at Operating Stress.* Apply the theory of tolerance intervals and use the tables of the one-sided Lower Tolerance Limits,  $LTL$ , of Natrella [6], to compute the minimum fatigue life of a full-size component,  $(minNf_3)_{op}$ , at the operating stress amplitude,  $(\sigma_a)_{op}$ , as a function of the sample size,  $n$ , the confidence level,  $\gamma$ , and the lack or “Failure” of coverage,  $Fp (=1 - p)$ .
- Step 5:** *Minimum Level 3 Life at Operating Stress and Extremely Low Failure of Coverage.* Using a nonlinear least squares fit algorithm and the physical assumption that the one-sided Lower Tolerance Limit ( $LTL$ ), at 95% confidence level, of the fatigue life, i.e., the minimum cycles-to-failure,  $minNf_3$ , of a full-size component, *cannot be negative* as the lack or “Failure” of coverage ( $Fp$ ), defined as  $1 - p$ , approaches zero, we estimate the minimum cycles-to-failure,  $minNf_3$ , at extremely low “Failure” of coverage,  $Fp$ , say, between  $10^{-3}$  to  $10^{-7}$ .

In Section 7, we extend our theory of a multi-scale fatigue life model to a probabilistic failure model by observing that the concept of coverage is closely related to that of an inspection strategy, and the predominant cause of failure of a full-size component is conceivably due to the “Failure” of inspection

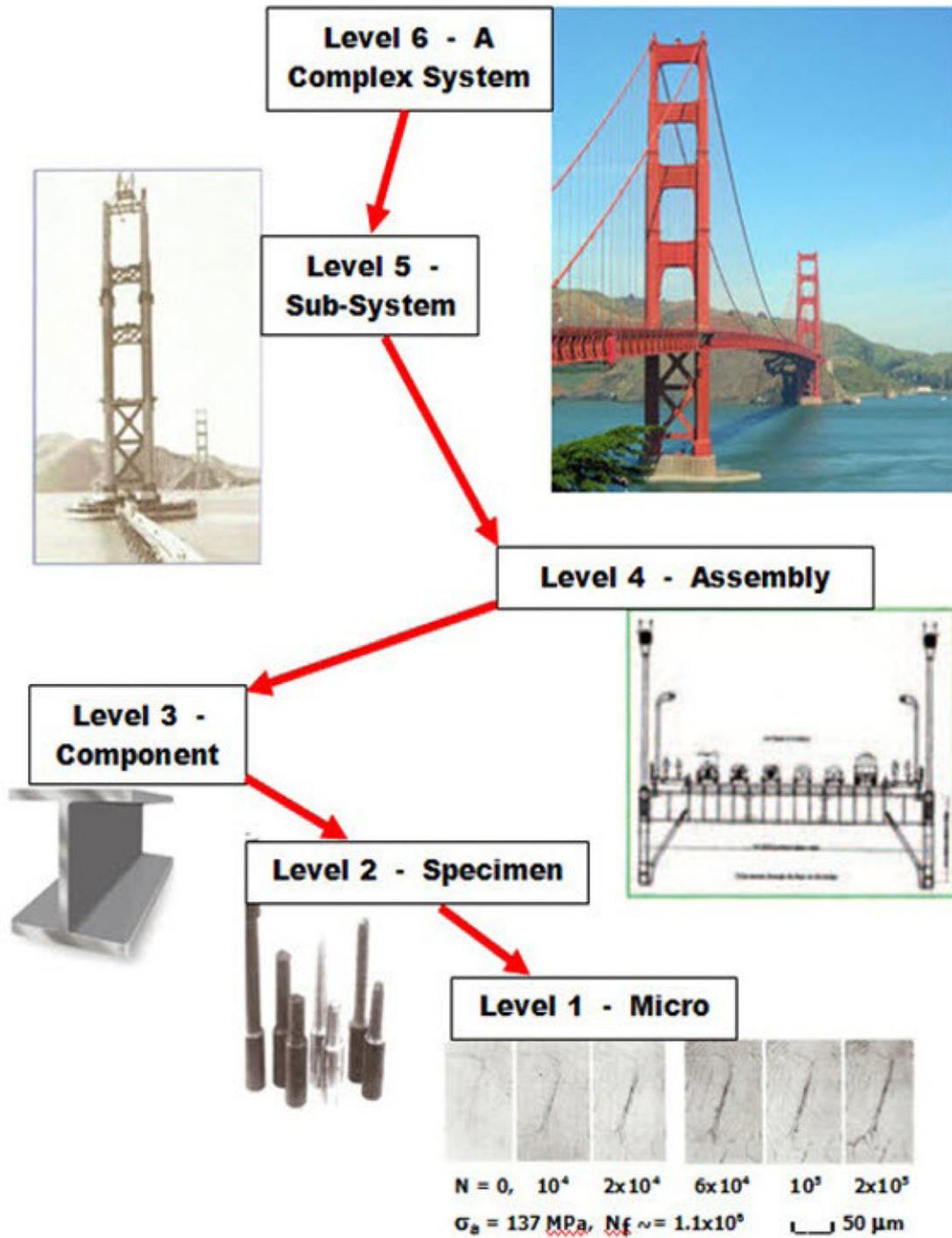


Fig. 3. A 6-level multi-scale representation of a complex structure such as a suspension bridge.

or coverage. This observation allows us to assume that the quantity,  $Fp$ , could be equated to a Failure Probability,  $FP$ , thus creating a new approach to estimating the frequency of in-service inspection of a full-size component. To illustrate this new approach of linking a fatigue model with a risk analysis, we include a numerical example again using the published data of the fatigue of an AISI 4340 steel (Dowling, 1973 [7], and 1999 [8]) to generate the necessary uncertainties for ultimately performing a dynamic risk analysis, where it is feasible to produce a graphical plot of an estimate of risk with uncertainty vs. a predicted most likely date of a high consequence failure event becomes available for a risk-informed inspection strategy associated with a nuclear powerplant equipment.

Significance and limitations of this new approach to life modeling and inspection of full-scale components are presented and discussed in Section 8. An interesting new result appears in Section 8 when we apply a nonlinear least squares logistic function fit algorithm to the fatigue data and obtain a clearly defined endurance limit as well as a statistical distribution of the ultimate tensile strength. Some concluding remarks and a list of references are given in Section 9.

To facilitate readers in computing the uncertainty quantities associated with a standard linear least squares fit algorithm, we provide in Appendix a full listing of a computer code written in an open-source language named DATAPLOT [9,10].

## 2. Theory of a multi-scale fatigue life model. Step 1. (Level 2 life formula)

Almost all experiments in a fatigue testing laboratory began with the fabrication of test specimens, and there is an abundant literature on experimental fatigue data and empirical models using cycles-to-failure or time-to-failure at the specimen level (Level 2) as a fundamental quantity of measurement. That is why we begin the first step of our modeling effort by searching the literature and identify a single Level 2 formula for developing a multi-scale model.

In this paper, we will identify three such formulas, and choose the simplest one for subsequent implementation of a numerical example. As shown below in equation (3), the simplest one we chose came from a book by Dowling [8, p. 364], where the number of cycles of a constant-amplitude fatigue fracture failure,  $N_f$ , and the applied stress amplitude,  $\sigma_\alpha$ , are in a power-law relationship:

$$\sigma_\alpha = A(N_f)^B, \quad \text{or equivalently, } N_f = (\sigma_\alpha/A)^{1/B}, \quad (3)$$

where  $A$  and  $B$  are two empirical material property parameters that can either be estimated with uncertainties from a linear least squares fit of a set of  $\log(N_f)$  vs.  $\log(\sigma_\alpha)$  data, or obtained from material properties handbooks and databases for specific materials.

A more complicated one, as shown below in equation (4), also due to Dowling [8, p. 520], is for the prediction of the number of elapsed cycles to a constant-amplitude fatigue fracture failure,  $N_{if}$ , as a function of the cyclic stress range,  $\Delta S$ ,

$$N_{if} = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C(F\Delta S\sqrt{\pi})^m(1-m/2)} \quad (m \neq 2), \quad (4)$$

where  $m$  and  $C$  are two empirical material property parameters,  $F$  is a specimen-geometry-dependent constant, and  $a_f$  and  $a_i$  are two geometric parameters associated with the final and initial length, respectively, of a crack embedded in a test specimen.

A third example, as shown below in equation (5), which appeared in a paper by Fuller et al. [11] for a class of glassy materials, is for the prediction of the mean time-to-failure,  $MTTF$ , or,  $t_f$ , as a function of

the applied stress,  $\sigma$ ,

$$t_f = \frac{\lambda}{N' + 1} \left( \frac{S}{S_v} \right)^{N'-2} \sigma^{-N'}, \quad (5)$$

where  $\lambda$ ,  $N'$ ,  $S$ , and  $S_v$ , are four material property parameters to be determined from 3 sets of experimental data as explained in a companion paper by Fong et al. [12]. A brief explanation of the 4 material property parameters and one stress variable is given below:

$S$  is the initial strength

$S_v$  is the strength of an intended reference set of specimens

$\sigma$  is the tensile stress in the component

$\lambda$  and  $N'$  are constants from environmentally enhanced crack growth

In general, we aim to develop a multi-scale modeling approach to estimating the *minimum* time-to-or cycles-to-failure of a full-scale component made of any material as long as there exists a closed-form specimen level life formula. In order for the rest of our 5-step modeling effort to work, such formula is expected to assume the following generic form:

$$N_f, \text{ or, } t_f = f(\sigma_1, \sigma_2, \dots, \sigma_k; m_1, m_2, \dots, m_q; g_1, g_2, \dots, g_j), \quad (6)$$

where the cycles-to-failure,  $N_f$ , or the time-to-failure,  $t_f$  may depend on  $k$  number of stress variables,  $\sigma_1, \sigma_2, \dots, \sigma_k$ ,  $q$  number of material property parameters,  $m_1, m_2, \dots, m_q$ , and  $j$  number of geometric parameters,  $g_1, g_2, \dots, g_j$ . Furthermore, all of the  $k + q + j$  parameters or variables are either determined from experiments or specified by the user after consulting the literature.

### 3. Theory of a multi-scale fatigue life model. Step 2. (Level 2 life data)

After a Level 2 (specimen) life formula is identified (Step 1), we begin our Step 2 by either running fatigue experiments to obtain cycles-to-failure,  $Nf_2$ , as a function of the applied stress amplitude,  $\sigma_a$ , or compute  $Nf_2$  using the formula identified in Step 1 with the material property parameters in the formula estimated from either available experiments or handbooks.

To illustrate our modeling steps by a numerical example, we choose to work with finding the minimum cycles-to-failure of a critical nuclear power plant component made of an alloy steel named AISI 4030. Its Level 2 life formula is a power-law relationship as shown in equation (3). The fatigue experimental data for that material (after Dowling [7,8]) are listed in Table 1.

### 4. Theory of a multi-scale fatigue life model. Step 3. (Level 2 life with uncertainty)

In Step 3, we apply a standard linear least squares fit algorithm (see, e.g., Draper and Smith [13]) to obtain first a log–log plot of  $Nf_2$  vs.  $\sigma_a$ , as shown in Fig. 4, and then an estimate of the predicted fatigue life,  $(Nf_2)_{op}$ , and its standard deviation,  $(sdNf_2)_{op}$ , for some operating stress amplitude,  $(\sigma_a)_{op}$ , as shown in Fig. 5. We assume in our numerical example that the operating stress amplitude,  $(\sigma_a)_{op}$ , is 398 MPa, with the corresponding value of the quantity,  $\log_{10}\{(\sigma_a)_{op}\}$ , equal to 2.60. A complete listing of a computer code that solves the linear least squares fit with uncertainty problem and is written in an open-source language named DATAPLOT [7,8], is given in the Appendix.

Table 1  
Fatigue data for AISI 4340 Steel (Dowling [7,8])

Stress amplitude $\sigma_a$ , MPa	Cycles-to-failure $Nf_2$ , Cycles
948	222
834	992
703	6,004
631	14,130
579	45,860
524	132,150

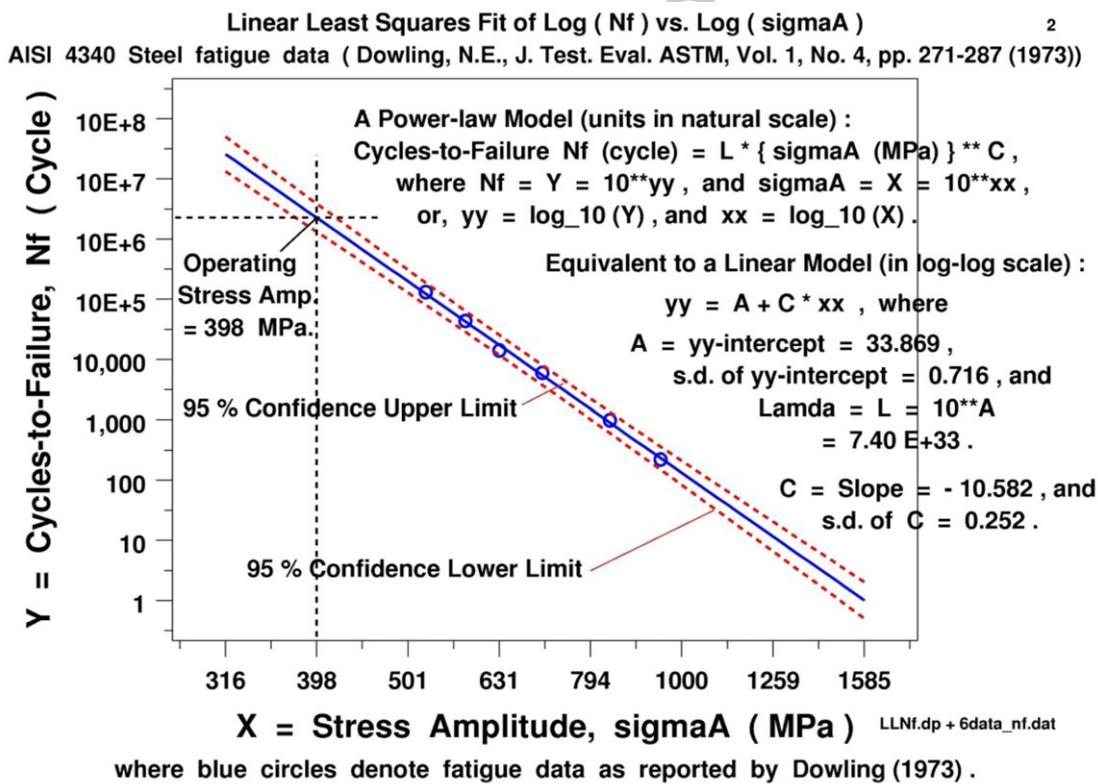


Fig. 4. A linear least squares fit of six fatigue specimen data (after Dowling [7,8]).

**5. Theory of a multi-scale fatigue life model. Step 4. (Level 3 minimum life)**

In Step 4, we apply the statistical theory of tolerance intervals (see, e.g., Nelson et al. [4]) and use the tables of the one-sided Lower Tolerance Limits, *LTL*, of Natrella [6], to compute the minimum fatigue life of a full-size component,  $(minNf_3)_{op}$ , at the operating stress amplitude,  $(\sigma_a)_{op}$ , as a function of the sample size,  $n$ , the confidence level,  $\gamma$ , and the lack or “Failure” of coverage,  $Fp (=1 - p)$ . The result of our calculations is given in Table 2.

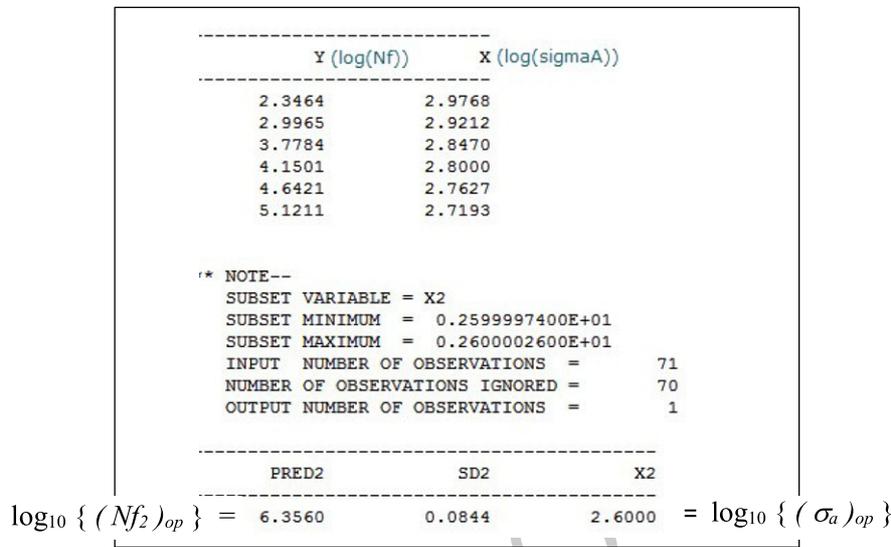


Fig. 5. A screen output of a computer code named “LLUQ.dp” (Appendix).

Table 2  
One-sided LTL vs. (1 - p) between p = 0.75 and 0.999 for n = 6 and γ = 0.95

	Confidence Level, γ, = 0.95				
	0.75	0.90	0.95	0.99	0.999
Coverage, p	0.75	0.90	0.95	0.99	0.999
Lack or “Failure” of coverage $Fp = 1 - p$	0.25	0.10	0.05	0.01	0.001
For n = 6					
From Natrella [6] K	1.895	3.006	3.707	5.062	6.612
From Step 3 $(Nf_2)_{op}$	2.26986 E+6	2.26986 E+6	2.26986 E+6	2.26986 E+6	2.26986 E+6
From a special computational procedure given in equation (7)** $(sdNf_2)_{op}$	0.44112 E+6	0.44112 E+6	0.44112 E+6	0.44112 E+6	0.44112 E+6
$K^*(sdNf_2)_{op}$	0.83592 E+6	1.32601 E+6	1.63523 E+6	2.23295 E+6	2.91669 E+6
$(minNf_3)_{op} = one-sided$	1.43393 E+6	0.94385 E+6	0.63463 E+6	0.03691 E+6	-0.64683 E+6
$LTL = (Nf_2)_{op} - K^*(sdNf_2)_{op}$					

\*\*The estimation of the standard deviation of  $(Nf_2)_{op}$  from a log–log plot of  $Nf_2$  vs.  $\sigma_a$  requires a special computational procedure as described below:

From Fig. 5, we obtain  $\log_{10}[(Nf_2)_{op}] = 6.3560$ , and  $sd\{\log_{10}[(Nf_2)_{op}]\} = 0.0844$ . From the statistical theory of error propagation (see, e.g., Ku [14]), we find a closed-form relationship between the standard deviation of  $\log_e(Nf)$ , or,  $sd\{\log_e(Nf)\}$ , and  $sd(Nf)$  as follows:  $sd\{\log_e(Nf)\} = \{sd(Nf)\}/Nf$ . Since  $\log_e(Nf) = \log_e 10 * \log_{10}(Nf)$ , we now have  $\log_e 10 * sd\{\log_{10}(Nf)\} = \{sd(Nf)\}/Nf$ , and,

$$(sdNf_2)_{op} = \log_e 10 * sd\{\log_{10}(Nf_2)\} * Nf_2 = 2.30259 * 0.0844 * 2.26986E + 6 = \mathbf{0.44112E + 6}. \quad (7)$$

Table 3  
One-sided *LTL* vs.  $(1 - p)$  between  $p = 0.75$  and  $0.95$  for  $n = 6$  and  $\gamma = 0.95$

	Confidence Level, $\gamma, = 0.95$				
	0.75	0.80	0.85	0.90	0.95
Coverage, $p$	0.75	0.80	0.85	0.90	0.95
Lack or "Failure" of Coverage $Fp = 1 - p$	0.25	0.20	0.15	0.10	0.05
For $n = 6$					
From Natrella [6] $K$	1.895	2.265 <sup>#</sup>	2.635 <sup>#</sup>	3.006	3.707
From Step 3 $(Nf_2)_{op}$	2.26986 E+6	2.26986 E+6	2.26986 E+6	2.26986 E+6	2.26986 E+6
See Step 4, equation (7) $(sdNf_2)_{op}$	0.44112 E+6	0.44112 E+6	0.44112 E+6	0.44112 E+6	0.44112 E+6
$K^* (sdNf_2)_{op}$	0.83593 E+6	0.99914 E+6	1.16235 E+6	1.32601 E+6	1.63523 E+6
$(minNf_3)_{op} = (Nf_2)_{op} - K^* (sdNf_2)_{op}$	<b>1.43393 E+6</b>	<b>1.27072 E+6</b>	<b>1.10751 E+6</b>	<b>0.94352 E+6</b>	<b>0.63463 E+6</b>

<sup>#</sup>Values of  $K$  for  $p = 0.80$  and  $0.85$  are obtained by interpolating tabulated values in Natrella [6].

## 6. Theory of a multi-scale fatigue life model. Step 5. (Life at small failures of coverage)

It is interesting to observe that in the last Step 4, an estimate of the quantity,  $(minNf_3)_{op}$ , at small "Failure" of coverage,  $Fp$ , say,  $0.001$ , turns out to be negative. This is physically meaningless, because the fatigue life of an engineered product cannot be negative. In this final Step 5, we first ignore the estimates of  $(minNf_3)_{op}$  at low  $Fp$  such as  $0.01$  and  $0.001$ , and re-calculate  $(minNf_3)_{op}$  at a reasonable range of  $Fp$ , namely, between  $0.25$  and  $0.05$ , to obtain a revised result of Table 2 as shown in Table 3.

We then use a nonlinear least squares algorithm and a 3-parameter logistic function (see, e.g., Fong et al. [15–17]) to fit the five data points of  $(minNf_3)_{op}$  vs.  $Fp$  in Table 3 with the assumption that the one-sided Lower Tolerance Limit (*LTL*), at 95% confidence level, of the fatigue life, i.e., the minimum cycles-to-failure,  $(minNf_3)_{op}$ , of a full-size component approaches zero as the lack or "Failure" of coverage ( $Fp$ ), defined as  $1 - p$ , approaches zero. This nonlinear fit allows us to estimate  $(minNf_3)_{op}$  at extremely low "Failure" of coverage,  $Fp$ , say, between  $10^{-3}$  and  $10^{-7}$ . The result is shown in Fig. 6, and this completes our 5-step multi-scale fatigue life modeling of a full-size component or structure.

In Figs 7 and 8, we show the results of applying the multi-scale model to a real-life situation, where a nuclear powerplant rotary equipment made of AISI 4340 steel is designed to run at 60 RPM for 25% of continuous runtime at the operating stress amplitude of 398 MPa. We also assume that the lack or "Failure" of coverage,  $Fp$ , can be equated to a failure probability,  $FP$ , with the implication that our multi-scale model can become a probabilistic failure model as basis for a risk analysis.

## 7. From a multi-scale fatigue model to a dynamic risk analysis of inspection frequency

In this section, we extend our theory of a multi-scale fatigue life model to a probabilistic failure model by observing that the concept of coverage is closely related to that of an inspection strategy, and the predominant cause of failure of a full-size component is conceivably due to the "Failure" of inspection or coverage. This observation allows us to assume that the quantity,  $Fp$ , could be equated to a Failure

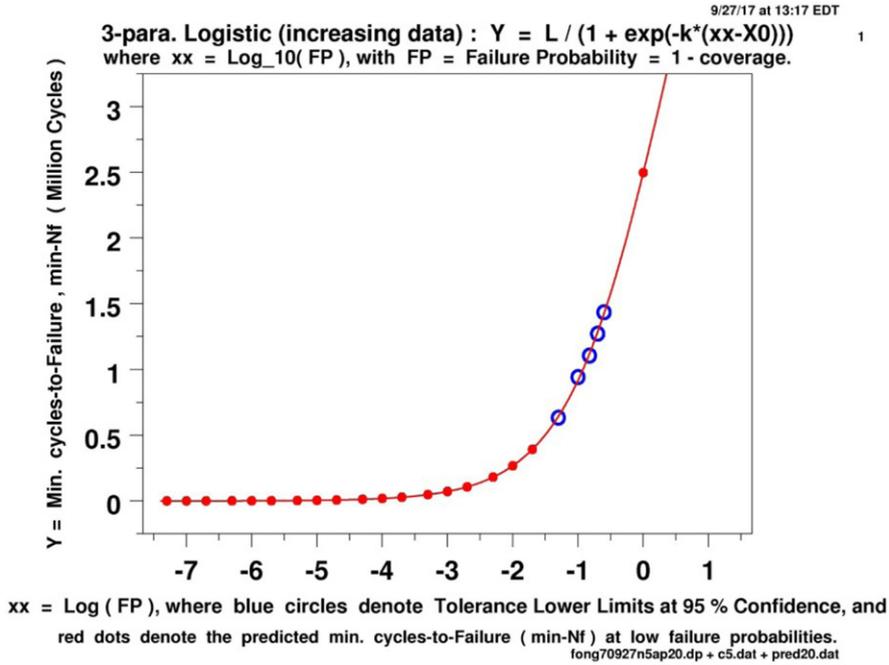


Fig. 6. A nonlinear least squares fit of five Lower Tolerance Limit data.

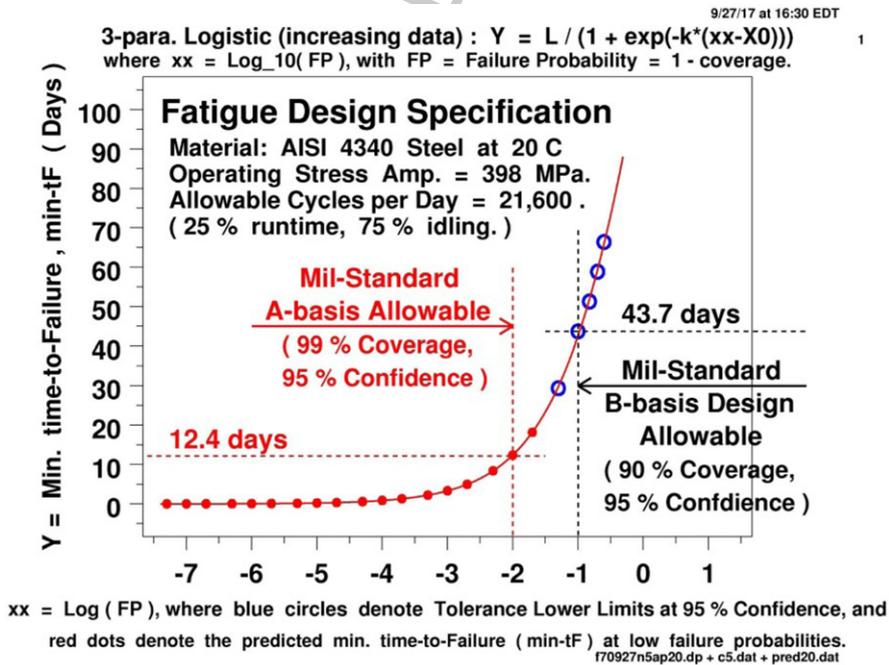


Fig. 7. Predicted minimum time-to-failure vs. “Failure” of coverage or failure probability.

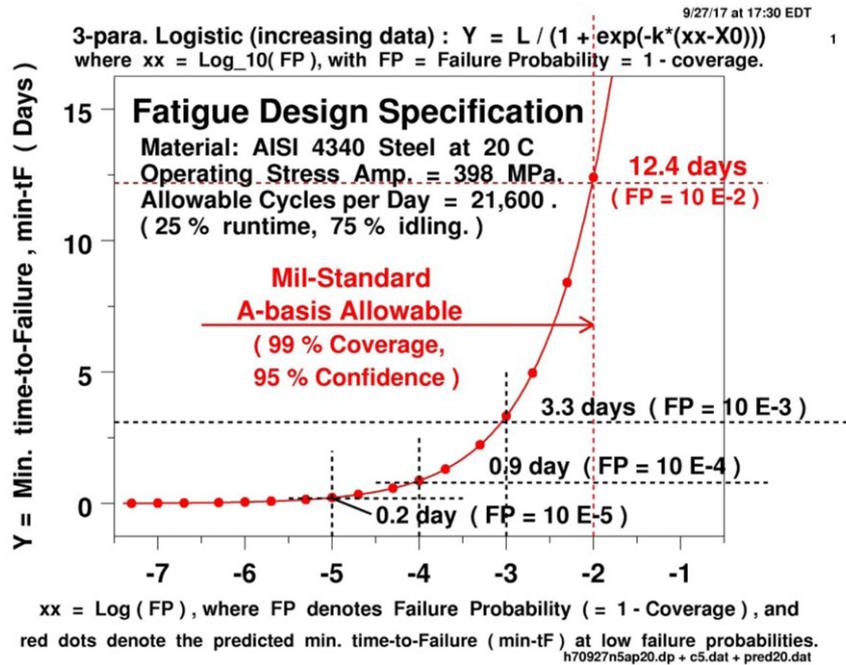


Fig. 8. Predicted minimum time-to-failure at very low failure probabilities.

Probability,  $FP$ , thus creating a new approach to estimating the frequency of in-service inspection of a full-size component.

To illustrate this new approach of linking a fatigue model with a risk analysis, we continue our numerical example on the prediction of a time-to-failure (days) of a critical nuclear powerplant equipment vs. failure probability,  $FP$ , as shown in Fig. 8. Assuming that the consequence of an accident due to the failure of that equipment varies from a low of \$10 million to a high of \$100 million with a median of \$50 million, and accepting the validity of the simple equation that risk is the product of failure probability and consequence, we arrive at a graphical plot, as shown in Fig. 9, of an estimate of risk with uncertainty vs. a predicted most likely date of a high consequence failure event at a nuclear powerplant. This plot, and similar ones for other critical components, can become a valuable tool for a risk-informed inspection strategy associated with the maintenance of a nuclear powerplant.

## 8. Significance and limitations of the multi-scale fatigue life model

Statistical methods and concepts have been known to and applied by workers in fatigue for at least 70 to 80 years. A 1977 review of the literature by Harter [18] on the specialized topic of the size effect on material strength alone, for example, listed about a thousand papers. The subject of a multi-scale fatigue life modeling based on measurement data and imaging at microscopic, specimen, and component levels was addressed by the first author [19] in 1979 with a concluding remark that said,

*“...There is a qualitative difference between the use of statistical tools in mechanism research and that in fatigue specimen and component life testing.”*

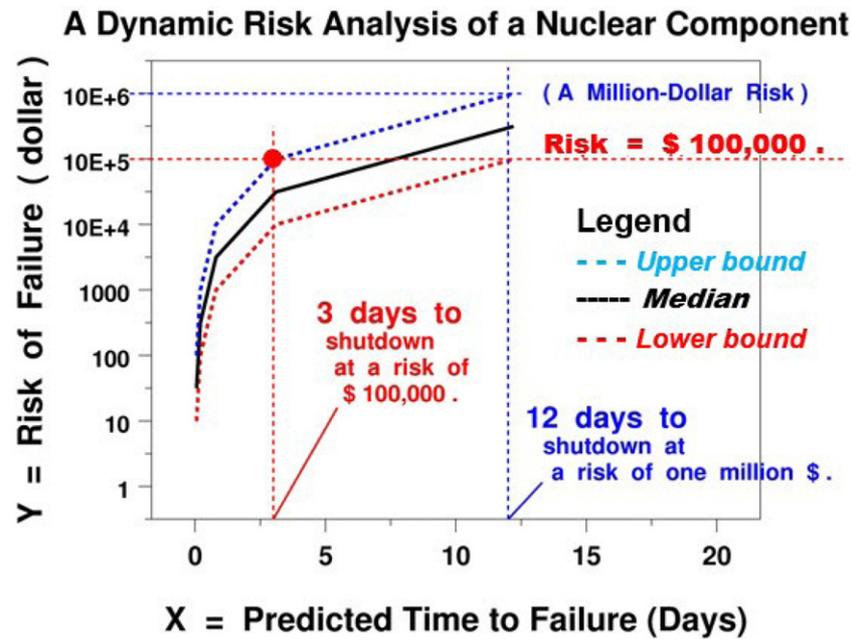


Fig. 9. A dynamic risk analysis of a nuclear component made of an AISI 4340 alloy steel.

The model presented in this paper clearly belongs to the second category. Nevertheless, the idea of using quantitative information at one level, say, Level 1-Micro, to predict fatigue life with uncertainty quantification at a higher level, say, Level 2-Specimen, at extremely high coverage or, equally plausible, high reliability, is generic. The modeling methodology presented in this paper is, therefore, *significant* not only to the advancement of knowledge in the second category, but also in the first, namely, fatigue mechanism research, where a huge amount of information is available at Level 1-Micro, and life prediction at Level 2-Specimen may similarly be modeled with uncertainty quantification.

The multi-scale life model presented in this paper is also new and significant, because for the first time, a physical assumption on the impossibility of a negative life at extremely high coverage has been made to extract from the model new life predictions that are useful to planning inspection of critical components. For high consequence systems with very low failure probability events, a credible risk analysis is generally very difficult because of the lack of data at low failure probabilities. The results of our 5-step multi-scale model should help engineers in making better risk-informed design and maintenance decisions.

However, the proposed model does have limitations that need to be discussed. First of all, the use of the one-sided lower tolerance limit tables of Natrella [6] is strongly linked to the assumption of a normal distribution for the fatigue life. Recent work by Fong et al. [20] on relaxing the normality assumption to include 2-parameter Weibull, 3-parameter Weibull, 2-parameter Lognormal, and 3-parameter Lognormal, should be capable of addresses that shortcoming. Secondly, the proposed model uses the linear least squares fit algorithm to estimate the mean and standard deviation of life at a given operating stress or stress amplitude, for which no experimental data is available because it will take too long to do such an experiment. This shortcoming turns out to make the model more conservative, as shown in Figs 10 and 11, where we re-plot the linear fit in Fig. 10 with  $\log(\text{stress})$  vs.  $\log(\text{life})$  and produce in Fig. 11 a nonlinear least square fit (see, e.g., Fong et al. [17]) with the linear line superposed on the nonlinear line to show

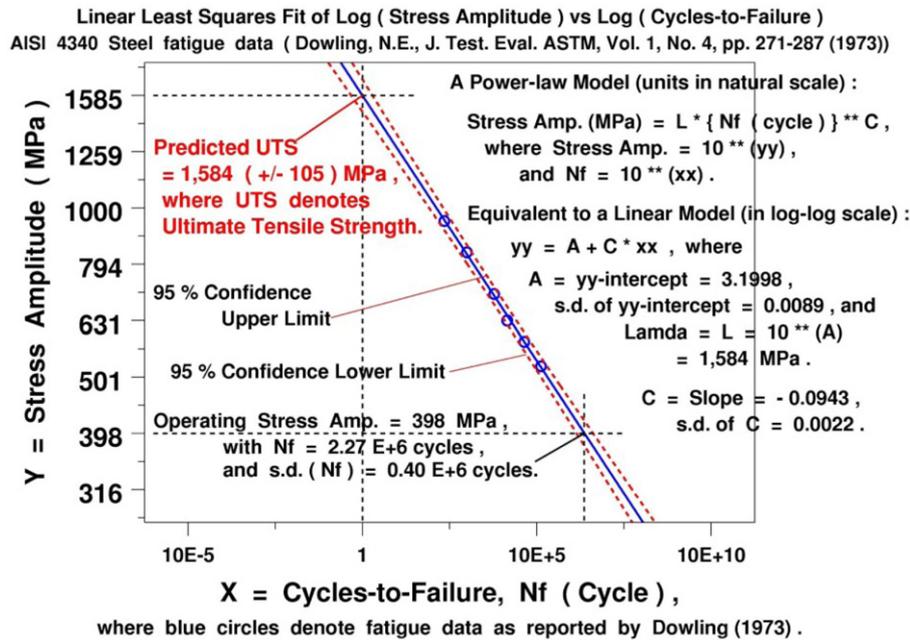


Fig. 10. An alternative linear least squares log–log plot of Stress vs. Life for Table 1 data.

why the linear model is more conservative. A third limitation is due to the assumption that low “Failure” coverage equates low failure probability. That assumption needs to be and could conceivably be validated when the results of our model are applied.

## 9. Concluding remarks

An uncertainty-based multi-scale fatigue life model has been presented with a numerical example using the 1973 published fatigue data of six specimens of an AISI 4340 alloy steel.

The modeling methodology is presented in five steps, with the first three describing the statistics and uncertainty quantification of Level 2, the specimen level, and the last two, that of Level 3, the component level. The effort of the first three steps is innovative, because it allows the modeler to estimate the uncertainty of the predicted Level 2 life at any operating stress or stress amplitude. The effort of the last two steps is also new, because it transforms the uncertainty of the predicted Level 2 life into that of the predicted Level 3 life with an added uncertainty due to a new statistical concept known as ‘coverage.’

The combined effort of the five modeling steps is to yield a predicted minimum life vs. failure of coverage or failure probability curve such that for the first time it is feasible for an engineer to predict minimum life at extremely low “failure” of coverage or failure probability between, say,  $10^{-3}$  and  $10^{-7}$ . This curve has been found to be useful to engineers when they are required to make risk-informed decisions on operation and maintenance.

The statistical modeling and analysis methodology of fatigue data presented in this paper also led us to a new result when we applied, instead of the classical linear regression fit, a nonlinear least squares 4-parameter logistic function fit. As shown in Fig. 11, a nonlinear regression gave us not only a prediction of the statistical endurance limit, but also that of a statistical distribution of the ultimate tensile strength.

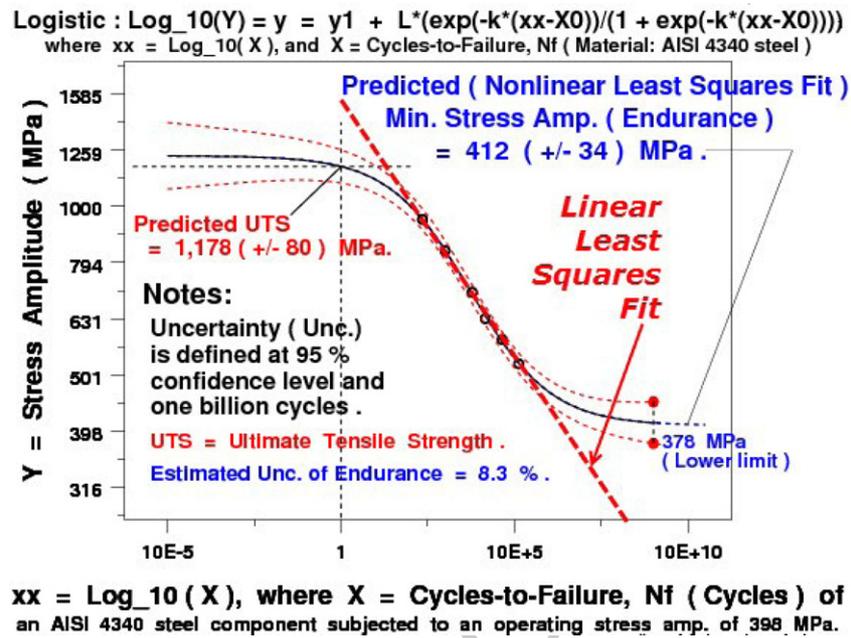


Fig. 11. A nonlinear least squares curve superimposed with a linear curve for the same data of the fatigue of an AISI 4340 alloy steel (after Dowling [7,8]).

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The first author (J.T.F.) wishes to thank the late Professor JoDean Morrow of the University of Illinois, Urbana, IL, for his contribution to a number of discussions he had with the first author in the 1970s on solving a multi-scale fatigue life modeling problem. We are also indebted to Professor Norman Dowling of the Virginia Polytechnic and State University for using his 1973 published fatigue data of an AISI 4340 alloy steel in our paper as the basis for an example to illustrate our theory.

**Disclaimer**

Certain commercial equipment, instruments, materials, or computer software is identified in this paper in order to specify the experimental or computational procedure adequately. Such identification is not intended to imply recommendation or endorsement by the U.S. National Institute of Standards and Technology, nor is it intended to imply that the materials, equipment, or software identified are necessarily the best available for the purpose.

**Appendix: Listing of a DATAPLOT code on a linear regression with uncertainty band**

```

=====
.   This is a DATAPLOT program.           Filename: LLUQ.dp
.
.   Note:      To run this code, we need the following additional macros:
.
.               Data file:                6data_nf.dat
.
.               Annotation file:  annotate.dp
=====
.   Purpose:  Linear Least Squares Fit (LLSF, or, LL) of Yvs. X
.             data in a log--log plot with xx = log(X), yy = log(Y),
.             and a linear model: yy = A + C * xx, with a 95%
.             confidence bounds as uncertainty quantification (UQ).
.
.   Reference:  Draper, N. R., and Smith, H., Applied Regression
.             Analysis. Chap. 1-3, pp. 1-130. Wiley (1966).
=====
.   Date first coded: December 8, 2005           Revised: Sep. 17, 2017
.
.   by:        Jeffrey T. Fong, Applied & Computational Mathematics Division
.   and        N. Alan Heckert, Statistical Engineering Division
.             NIST, Gaithersburg, MD 20899, U.S.A.
.
.   Contact:   fong@nist.gov, or, fong70777@gmail.com, (301) 975-8217
=====
.-----1.0 Preliminary Commands-----
probe iopsy1
if probeval = 1
    device 1 x11
end of if
device 2 ps
device 2 color on
.
let string header = Linear Least Squares Fit of Log (Nf) vs. Log (sigmaA)
let string trailer = LLNf.dp + 6data_nf.dat
let string currdate = 9/17/2017 10:30 EDT fong@nist.gov
let pagecoun = 0
=====
.-----1.1 Read input data file-----
skip 15
read          6data_nf.dat          y1          x1
skip 0
=====
.-----1.2 Convert data to log_10 format-----
let x = ln(x1)/ln(10.0)
let y = ln(y1)/ln(10.0)
=====

```

```

.-----1.3 Add graphics commands for double plots-----
let voffset = .4
character circle blank
character hw 1.5 1.1
character color blue blue
character thickness 0.3 0.3
.
. char offset 0 voffset all
lines blank dash
.
justification center
hw 3 1.5
color black
move 50 92
text Y= Nf, yy = Log(Y). X = sigmaA (MPa), xx = Log(X).
.
.=====
.-----1.4 Specify xlimits ylimits graphics commands--
.
xlimit 2.5 3.2
xtic label size 3
.
ylimit 0 8
ytic label size 3
.=====
.-----1.5 Specify xlabel ylabel graphics commands----
xlabel size 4
xlabel X = Stress Amplitude, sigmaA (MPa)
.
x3label size 3
x3label where blue circles denote fatigue data as reported by Dowling (1973).
.
xtic label format alphabetic
xtic label contents 316 398 501 631 794 1000 1259 1585
.-----
ylabel size 4
ylabel Y= Cycles-to-Failure, Nf (Cycle)
.
ytic label format alphabetic
ytic label contents 1 10 100 1,000 10,000 10E+5 10E+6 10E+7 10E+8
.
ylabel displacement 11
.=====
.-----1.6 Add annotation commands for table of data-----
justification left
hw 4 2
color black
move 42 82

```

```

text Table of Fatigue Data (Dowling, 1973)
.
hw 3.2 1.6
move 54 77
text sigmaA, MPa      Nf, cycles
.
move 60 73
text 948              222
.
move 60 69
text 834              992
.
move 60 65
text 703              6,004
.
move 60 61
text 631              14,130
.
move 60 57
text 579              43,860
.
move 60 53
text 524              132,150
.
=====
.-----1.7 Plot Log(Y) vs. Log(X), or, y vs. x-----
plot y x
call annotate.dp
.
=====
.          END OF PLOT 1      END OF PLOT 1      END OF PLOT 1
.
=====
.
.          step 2: fit the data via model (straight line)
.
=====
.-----2.1 Linear least squares fit and output analysis result in a file---
echo on
.          capture screen on
.          capture 6data_nf.out
.
let function g = a + c*x
fit y = g
.
.          end of capture
echo off
.
=====
.-----2.2 Save 4 analysis result parameters on y-intercept and slope---
skip 0
read parameter  dpst1f.dat      int      sdint

```

```

.
skip 1
read parameter dpstif.dat slope sdslope
.
print int sdint slope sdslope
.=====

.-----2.3 calculate y-intercept a, lamda, and their stand. dev.-----
let a4 = round(a, 4)
. print a4
.
let lamda = 10**a
.
let sdint4 = round(sdint, 4)
. print sdint4
.
let sdlamda = sdint*lamda
.
.=====

.-----2.4 calculate slope c and its stand. dev.-----
let c4 = round(c, 4)
. print c4
.
let sdslope4 = round(sdslope, 4)
.=====

.-----2.5 Plot data with its linear least squares fit---
plot y x and
plot g for x = 2.5 0.1 3.1
call annotate.dp
.=====

.-----2.6 Add post-plot annotation commands-----
hw 2.8 1.4
move 50 92; just center
text AISI 4340 Steel fatigue data (Dowling, N.E., J. Test. Eval. ASTM, Vol.
1, No. 4, pp. 271-287 (1973))
.
hw 3 1.5
move 50 66; just left
text Equivalent to a Linear Model (in log-log scale):
.
move 60 61
text yy = A + C * xx, where
.
move 58 56; just left
text A = yy-intercept = 33.869,
.
move 62 52; just left

```

```

text s.d. of yy-intercept = 0.716, and
.
move 70 48; just left
text Lamda = L = 10**A
.
move 76 44
text = 7.40 E+33.
move 72 38; just left
text C = Slope = -10.582, and
.
move 76 34
text s.d. of C = 0.252.
.
move 32 84; just left
text A Power-law Model (units in natural scale):
.
move 32 80
text Cycles-to-Failure Nf (cycle) = L * {sigmaA (MPa)} ** C,
.
move 36 76
text where Nf = Y= 10**yy, and sigmaA = X = 10**xx,
.
move 38 72
text or, yy = log_10 (Y), and xx = log_10 (X).
.
=====
.           END OF PLOT 2           END OF PLOT 2           END OF PLOT 2
.=====
.
.=====
.           step 3: add 95% CL lower and upper limit lines
.=====
.-----3.1 Preliminary graphics and annotation commands-----

pre-erase off
.
let xxmax = 3.2
let xxmin = 2.5
.
ylim 0 8
.
let x2 = ^xxmin 0.01 ^xxmax
.=====
.-----3.2 Add uncertainty quantification code by calculating 95% confidence
.           Limit lines (see Draper & Smith, 1966, pp. 1-130)
.
.----- Note x = log_10(x1) and y = log_10(y1). x1, y1 are raw data.
let xbar = mean x

```

```

let del2 = x - xbar
let del2sq = del2**2
let denom = sum del2sq
.
let num = (x2-xbar)**2
let term3 = num/denom
.
let n = number y
.
let term2 = 1/n
let term1 = 1
.
let sd2 = ressd*sqrt(term1+term2+term3)
.
let df = n-2
.
let t = tppf(.975,df)
let half = t*sd2
.
let function g2 = a4 + c4*x2
.
let pred2 = g2
.
let lower2 = pred2 - half
let upper2 = pred2 + half
.
char circle blank blank blank
lines color black blue red red
lines thickness 0.1 0.3 0.3 0.3
lines blank solid dash dash
.
print y x
. pause
.
print pred2 sd2 x2 subset x2 = 2.6
pause
.
print pred2 sd2 lower2 upper2 x2
. pause
.=====
.-----3.3 Add plots of two 95% confidence limit lines-----
plot y x and
plot pred2 lower2 upper2 vs x2
.=====
.-----3.4 Add post-plot annotation commands-----
hw 3 1.5
just left
move 16 66

```

```

text Operating
.
move 15.5 62
text Stress Amp.
.
move 16 58
text = 398 MPa.
.
move 16 48
text 95% Confidence Upper Limit
.
lines solid
line thickness 0.1
lines color red
drawdddd          2.825  3.25      2.87   3.6
.
move 22 28; just left
text 95% Confidence Lower Limit
drawdddd          2.90   0.50      3.035  1.50
.
lines dash
line thickness 0.25
lines color black
drawdddd          2.6   -0.6      2.6    7.4
drawdddd          2.445  6.356     2.67   6.356
.
lines solid
line thickness 0.15
drawdddd          2.56   5.8       2.60   6.356
.
=====
.
      END OF PLOT 2a (same as Plot 2 with addition of 95% CL lines
.
=====
exit

```

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